EXPONENTIATION IS NOT REPEATED MULTIPLICATION: DEVELOPING EXPONENTIATION AS A CONTINUOUS OPERATION

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The concept of exponents has been shown to be problematic for students, especially when expanding it from the domain of positive whole numbers to that of exponents that are negative and later rational. This paper presents a theoretical analysis of the concept of exponentiation as a continuous operation and examines the deficiencies of existing approaches to teaching it. Two complementary theoretical frameworks are used to suggest an alternative definition for exponentiation and guiding principles for the development of a teaching trajectory, and then to analyze an example of the hypothetical learning of a student who goes through the first task in the trajectory. The paper concludes with some possible implications on curriculum and task design, as well as on the development of mathematical operations.

Keywords: Algebra and Algebraic Thinking; Instructional Activities and Practices

The Problem with Exponents

Research has found the concept of exponents problematic for both students and teachers of all levels (Confrey & Smith, 1994; Elstak, 2007; Goldin & Herscovics, 1991; Weber, 2002). The most common definition that students have for exponents is that of repeated multiplication, for example $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ which is two multiplied by itself five times. This limited view of exponentiation, although simple to understand, prevents students from understanding the behavior of exponents in non-natural-number powers. For example, if exponentiation is repeated multiplication then something to the power of zero might be seen as ambiguous—should it be zero or one? Moreover, there is no meaning to multiplying something by itself a negative number of times. An extension to fractional powers that preserves the sense of repeated multiplication is an impossible task, and often fractional exponents are presented as "a different way to write radicals" and connected to repeated multiplication in an artificial way. As they finish their high school unit on exponents, student end up not being able to look at exponentiation as a *continuous* process (in terms of the exponent value) and, in the best case scenario, have a few different models connected to one another through loose logic.

Although it is possible to develop the concept of exponentiation as a case-based operation, this approach may result in various negative implications, such as the perception that exponentiation is always an increasing operation, an inability to work with exponents as continuous functions later in the curriculum and difficulties in understanding the rate of change of an exponential function and its derivative. One other problem that is rarely addressed in the mathematics education literature lies in the fact that students do not have any qualitative sense of the changing growth rate of an exponential function that results in an inability to explain such ideas as compounding or population growth without having to calculate its value numerically.

Existing Approaches

There have been cases of teachers who tried to develop students' understanding of the exponents' rules through the process of proof and mathematical consistency by moving from one rule to another in deductive manner with the goal of allowing the students to see the connection between them. This, however, does not create a single view of exponents that students can work with across domains, and it always remains as a sequence of logical operations that explains the various cases of exponents (positive, negative, zero, and rational). The research community has made several attempts at developing a

conceptual understanding of the concept of exponents while at the same time aiming at building a single view of the operation across domains.

The Functional Approach

The teaching of algebra in a functional-based approach was first suggested by Goldin and Herscovics (1991), later to be tested by Elstak (2007) in a teaching experiment. In this approach, the understanding of negative, zero and rational exponents comes from constructing the definition of an exponential function starting from natural exponents, and later investigating this function to expand the notion to other domains. In Elstak's teaching experiment, some of his students were able to logically connect the different cases of exponents, but could not give a single definition for all of them.

Developing Algorithms

Weber (2002) suggested that students be presented with a description of an algorithm to compute exponents, which later they express formally. The students wrote the exponential expressions as products of factors, and then completed activities in which they debated about the nature of rational exponents. Although Weber explained the expected learning path using APOS theory (Dubinsky, 1991), it was not clear how the "debate" stage helped the students develop a conceptual understanding of rational exponents that aligned with their original definition of the operation.

Exponents as Splitting

Confrey and Smith (1994, 1995) suggested that exponents be developed through the idea of splitting. Basing their design on students' familiarity with the idea of fair share and splitting, they gave special attention to the rate of change of the function, which was not given by other researchers. They aimed at developing in students the multiplicative comparison between the sizes of the quantity at different stages, with a focus on a multiplicative rate of change as being fixed throughout the work with exponents, in contrast to the varying additive rate of change. This basis allows students to develop a comprehensive view of exponents of positive base with integer powers (positive, negative, and zero). To extend the domain to rational bases and exponents, they relied on the contrast between the counting and the splitting worlds and offered a logical explanation that accounts for rational exponents. In contrast to their initial work that relied on splitting, the expansion to rational exponents does not offer a cognitively intuitive model to work with, and the case-based view is left unresolved.

What Is Missing?

Although some of those studies contain valuable insights, each has its own drawbacks with regards to the development of a comprehensive conceptualization of exponentiation as a continuous operation relying on a single image. One shared limitation of all of the above approaches is that they begin working in natural numbers and give the students a limited concept of exponents, later to be expanded in one way or another. Expansion to the real number domain is accomplished through formal explanations, definition or logical reasoning, but is not based on the original image the students have. Even in the splitting world, which is an extremely powerful idea based on students' own experience, there is no real meaning to the exponent itself that accounts for both positive and negative values. It ends up being a logical process explaining to the students that in the new "multiplicative" world negative exponents result in division. Rational powers are similarly not well addressed in the splitting world ending up being understood empirically through the identification of patterns. It is extremely hard to think of doing three and a half split operations since the splitting is done on countable sets. In essence, trying to build students' understanding based on a limited domain (natural number) as a starting point requires that students adjust their definition of exponents for every new domain extension, sometimes conflicting with the original one they had, resulting in a disconnected set of definitions.

Another limitation of the approaches is an overuse of calculations as a way to "understand" exponents, indicating a move towards an empirical instead of conceptual understanding. This is evident in the

algorithmic and functional approaches, and even in the work based on splitting, there is still some emphasis on generating sample values in order to understand rate of change.

Theoretical Frameworks

One goal when designing the following teaching trajectory is that it will allow students to develop a coherent understanding of the concept of exponentiation as a continuous operation with no contradictions as the domain of the exponent expands from positive whole numbers, to negative whole numbers (and zero), to fractional exponents, and finally to all real numbers. In order to design such a trajectory and to analyze its usefulness, I am using two different theoretical frameworks, which I argue to be complementary (Simon, 2009), and offer principles for the arrangement of the content and provide constructs that explain the learning of the student.

For the overall content organization approach and the analysis of the core understandings that are the foundation of the concept of exponents, I used the design principles as laid out by Davydov (1975). One of Davydov's main points is that the material should be organized, moving from the general to the specific. That is, the students should move from learning about the concept in its most general form and later to work through different cases which are manifestations of it in such a way that they develop a complete understanding of the concept. A principal goal in designing the teaching trajectory was to develop an image of exponents that students can repeatedly use in different cases without ever having to deal with contradictions. This is significantly different than building on the specific case of positive whole numbers and later expanding it to negative and fractional exponents.

A guiding example of the development of a concept from general to specific is the development of multiplication (Davydov, 1992). Davydov asserted that multiplication should not be viewed as repeated addition, raising similar issues to the ones described earlier with exponentiation being viewed as repeated multiplication. In his measurement-based approach, he distinguished between the way one *calculates* the value of a multiplication expression and the *image* of multiplication. Creating an image of multiplication as a change of unit of measurement allows Davydov to develop with his students a more general view of multiplication that also includes multiplication of fractions. My work builds on the previous work not because it is intended to address the issue of teaching multiplication, but because this example is used to demonstrate how it is possible to create a single image of a concept that can be used across cases.

However, Davydov's framework cannot be used to analyze the learning of the students from a cognitive perspective, and to complement it I used Simon et al's (2010) framework for learning through activity, building on Piaget's (2001) concept of reflective abstraction. This framework seeks to explain the transition from the point at which a student did not understand something to the point at which he or she does understand it (p. 77). The following are three principles I used from Simon et al's framework: the importance of a learner's activity, the importance of the learner's reflection, and the distinction between reflective abstractions and empirical learning processes (p. 74). I also note the goal-directed nature of the student's activity, as well as the development of the logical necessity that brings the student to anticipate the relevant results.

Principles in Developing the Concept of Exponents

The development of a single image of exponentiation is the key to developing a *continuous* concept of exponents, including natural and rational bases, as well as real numbers as exponents. The following are guiding principles for designing a trajectory that leads to such concept:

Separating the Image and the Calculation

Students should be able to consider the calculation of any exponential expression separately from the image of exponentiation. Whereas the calculation of positive, negative and rational exponents may involve different operations, creating a single image of exponents will allow them to reason about those as one concept and not as a disconnected set, and carry the same image they expand their domain of operation.

Beginning with a Qualitative Understanding of Exponents

One of the challenges in working with exponents is that the resulting values of the operation quickly become too big (or too small). This does not allow students to develop a real sense of the rate of change since they need to manipulate large numbers at the same time. Beginning with a qualitative (vs. quantitative) experience of exponents that builds on a visual representation will allow the students to build an understanding of the operation's effects and develop an expectation for the result of using different bases, without ever calculating the numerical value of the operation.

Using Physical Quantities of a Continuous Nature

In order to develop in students the understanding of exponents in the integer and rational domains, students should not be limited to working with discrete quantities. Quantities such as length, area, and volume allow them to perform the operation of exponentiation using any value for the exponent and experience the change in a continuous way. Moreover, these quantities are also measureable and visible so they lend themselves to the use of multiple representations.

Using Whole Numbers as a Case within the Continuous Domain

Beginning the calculations of exponents with whole numbers is a way for students to find the result of the operation. The suggested approach introduces whole numbers only once the *image* of exponents is established and without changing it moves to whole numbers as a particular case of *calculating* the result.

Building on Students' Intuitive Models

Confrey and Smith used a concept from the students' world that can be used intuitively as a basis for developing the mathematical concept, and it provides a strong foundation and an important entry point for the teaching trajectory.

Applying several of these principles together poses a great challenge. Continuous models in nature (e.g., temperature change, decay of matter) tend not to be directly measurable which means they cannot be used in a physical activity. The other models are mostly discrete (similar to the ones shown in the splitting approach). An activity that uses a quantity such as length and requires the students to change its size by, for example, stretching it, does not result in an exponential change. In order to create a true exponential change in a quantity, the students have to consistently change the power/speed they use while stretching the quantity, which is not a natural thing to do. In order to overcome such difficulties the suggested teaching trajectory relies on a technological environment that allows the students to change the exponent *continuously* and observe the change in the quantity.

What Is Exponentiation?

The starting point for building a continuous view of exponentiation is accepting that repeated multiplication is only a way to *calculate* exponents in the case of whole numbers, and not the *image* of it. This was understood by previous researchers—Confrey and Smith, for example, developed the image of exponentiation as a splitting operation and repeated multiplication was a way to calculate the number of elements after a sequence of splits. Even though exponentiation is a multiplicative idea at its core, this idea is not enough for building a mental model that students can relate to as they expand the concept and learn to calculate it in different cases.

One of the main differences between addition, multiplication and exponentiation is that the last one is an operation which produces a changing growth/reduction rate. It is also the basis for understanding the extreme changes that happen when exponentiation is repeated.

The basis for understanding the changing rate of exponentiation is that the growth/reduction in exponentiation is always *relative* to the quantity it operates on. Building on that understanding, I define the following *image* of exponentiation as the goal for the trajectory:

Exponentiation is an operation that *changes* a given quantity *multiplicatively* based on the *current size* of the quantity.

A few notes with regards to this image of exponentiation:

1. Through defining exponentiation as a general changing operation, students are not limited to thinking that "exponentiation makes bigger" which is sometimes assumed from the definition of repeated multiplication. The image can be used in the cases of positive as well as negative and fractional exponents.
2. The multiplicative rate as the change factor allows for fractional rate bases, which is important for building the single view across bases that was the goal of the instruction.
3. There is an underlying assumption, although not explicit, that the students can think of a change as related to a given size, meaning a proportional change.
4. The question of calculating the value of the exponential expression is left unanswered in this definition. It is essential that the students have a single model to build on, and that they see the calculations as based on specific cases within that image.

Developing the image of exponentiation is a first step in helping the students to construct the different cases of this image (in the various domains for bases and exponents). The image of exponentiation is reinforced by the use of symbolic language in a more formal mathematical definition and model which includes the following components:

1. Q_0 – the quantity being operated on, at the initial stage (time=0)
2. a – the multiplicative rate of change (natural or fractional) in a given time unit
3. x – the change in time units (as related to the time of the initial stage)
4. Q_x – the resulting quantity at time=x (this can be at any point in time)
5. a^x – the accumulated multiplicative change of the initial quantity (Q_x divided by Q_0).

Representation Model

The use of multiple representations and the utilization of dynamic control and manipulation, made possible using technology, can have a positive impact on students' learning in algebra (Kieran, 2006) and these are incorporated as part of the representational model. In addition, the representation must allow the use of continuous quantities and make it possible for the students to engage with the image without relying on specific calculations.

The Context of the Problem

Based on the desired characteristics of the image of the concept of exponentiation as described earlier, it is essential that the change described in the problem is one that is based on the current size of the quantity being changed. The suggested context problem (other variations exist of course) is the following:

Magic caterpillars need to eat a certain amount of leaves. The length (amount) that each caterpillar needs to eat is proportional to the caterpillar's length. The eating transforms the caterpillar - they end up being as long as the length of the leaves they ate (for simplification they "eat" in straight lines). We will examine the changes in the length of the caterpillars.

The context was chosen to support the development of a complete view of exponentiation:

1. It provides freedom in the selection of the initial quantity to be operated on, as well as the introduction of various bases, in the form of different types of caterpillars that change differently, being more (or less) hungry
2. Its structure allows for the introduction of both growth and decay, since the proportion of the food they need might be smaller or bigger than the caterpillars themselves
3. In the general form of the problem there is no mention of any period of time in which the food needs to be consumed. It will be introduced as a "feeding cycle" later as the students move to the quantitative section in which they calculate the values of the growth. This also allows for the use

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of multiple-size feeding cycles, which is the basis for working with any rational exponents, and developing a continuous view of the operation

The Visual Representation

The representation includes two critical components:

(a) Dynamic graphical representation of the caterpillar. Using a narrow rectangle as a representation of the caterpillar allows the students to focus on the length as the relevant quantity. Also, using the length representation answers the need for a continuous quantity, which can grow or shrink dynamically. This eliminates the problem caused by using discrete properties.

(b) Horizontal time axis with a slide bar. The use of a slide bar represents a continuous view of the exponent value allowing the students to work in fractional and integer values. Offering a dynamic manipulation of one element, the use of a slide bar is a known representation for time progression which is familiar to students (consider YouTube for example).

Bringing all the pieces together, and using the elements of the mathematical model with the representation model and the problem context, we have the following:

1. Q_0 is the size of the caterpillar at the given initial state (when the observation begins)
2. "a" is the property of the caterpillar which defines how much food it eat at each time unit
3. "x" is the change in time represented by the slide bar. The location of the marker on the slide bar shows how far they are (and in which direction) from the initial state
4. Q_x is the size of the caterpillar (the rectangular bar) at any point in time (the slide bar)
5. a^x is the result of calculating (or predicting) the change from the initial to the current state

Analysis of the Hypothetical Learning

The affordances of the definition and model described above are best demonstrated through an analysis of a student's *hypothetical* learning process. What follows is a description of the *hypothesized* learning of a student who performs two activities from the first task in a teaching trajectory which is based on the principles above. A full description of the teaching trajectory and analysis of the hypothesized learning as for each step is beyond the scope of this paper.

During the first three activities of the trajectory (not detailed here), the students work with quantities (the length property of the caterpillars, with no particular numerical value but of a comparable magnitude) that change in either exponential or linear form and understand that exponential growth (or decay) is faster than linear one, once the quantity reaches a certain size. They can explain the relationship between the size of the caterpillar and its growth (or decay) and focus on the fact that the bigger the given quantity is, the bigger the change is. Also, students are introduced to the formal mathematical notation. They now move to the fourth activity.

Activity 4: Comparing Different Quantities

In this activity, the student works with two caterpillars with the same base for the exponent and compares their growth, in absolute (additive) and proportional terms. Working in the same base and the same time period with different-sized caterpillars focuses the student on the initial size of the caterpillars. In reflecting on his activity he is expected to understand the logical necessity of the initial quantity explaining the difference in the resulting quantity, building on his prior knowledge of the relationship between the quantity and the change, and knowing that the initial quantity is the only attribute in which the caterpillars vary. I describe now the steps in the activity that lead to this understanding.

In the first step of this process, the student works with one caterpillar and establishes its growth rate, as in previously activities. He does this by using the slide bar marked with units as before to change the time value and compare the resulting size. From this, he can see that the growth is based on a particular base rate. For example, he might observe that the caterpillar grows by a multiple of 3 for every time unit in the case of a base of 3. He does this by comparing the quantity after one time unit, with the quantity at the beginning, or the quantity after x+1 units, with that of after x units.

Once the growth rate is determined, another quantity of a bigger size is introduced and placed next to the original one. The student is told it needs to eat the same proportion as the previous one, and is asked to predict which one will grow more. The student is expected to develop an understanding that the growth of a caterpillar is proportional to its original quantity, and is assumed to know that when multiplying two numbers by the same factor, the larger number will result in a larger product. Building on these two, the student will anticipate that the larger caterpillar will grow more, because "3 times a bigger number is bigger." This understanding is powerful since the student learns about a relationship which is not dependant on whether the given quantity is of a whole or fractional size. At the end of this activity the student concludes that for an equal amount of time units and the same growth rate, the larger quantity will grow by a larger amount. It is important to note that the student generalizes about exponents *without* looking for a numerical pattern, and that the process he goes through is based on reflection about the general process and the understandings developed about the nature of this activity.

Activity 5: Moving Forward and Backward in Time, Only Until the Present

In this activity students can move the scroll bar forward and backward between the present time and a particular point in the future (just to avoid infinite growth/decay in the future). They begin with a quantity of a particular size and are asked to explain whether it is growing (eating more than its size) or shrinking (eating less than its size), based on their observation of the behavior. This should be a simple question for the students, knowing they completed the previous activities in which they learned that when a caterpillar eats more than its size, it grows every day (towards the future) and the opposite in decay. The students are then asked to explain whether the quantity grows bigger or smaller if you move forward in time. The students are then asked to move the bar to a random point on the time line and answer the following question "in order to reach this state and from what you know of the behavior of the caterpillar, would the quantity had to be bigger or smaller before this point in time." Students, building on their "forward" thinking and activity of moving the bar forward from before, see the logical necessity that in order to reach a certain quantity, when a caterpillar is growing, it had to be smaller before (and similarly for decay it had to be bigger before). They are then asked "Knowing that a caterpillar is 'growing' caterpillar, if you look back in time, would that caterpillar be bigger or smaller?" and "how would it look if you move forward in time?" Although this can be checked by the students through the representation, at this point they already see the necessity of the caterpillar being smaller, since it had to grow to reach the given size (in the case of growth). The students learn to anticipate that if the caterpillar is of a growth type ($a>1$), then when moving forward in time (increasing the exponent) the quantity grows, and when moving backward in time (decreasing the exponent) it becomes smaller.

At the end of those five activities, students develop the concept of exponentiation as a change which has a rate proportional to the current size of the quantity, as a factor of time. Also, students can explain the change in a quantity when moving both forward and backward in time, which will serve as a precursor for the development of zero and negative exponents as places on the time line. Moreover, the use of a continuous time line sets the stage for other "time points" which will not be whole numbers. Students never used particular values for quantities so they never calculated the exponential change, and that supports the development of a continuous view of exponentiation and keeping the image of exponentiation separate from the calculation process.

Developing Continuous Concepts in Mathematics

The development of mathematical operations such as addition, multiplication and exponentiation usually begins with the positive whole number case and expands later to negative and rational numbers. I began this paper presenting the implications such an approach might have on the student and suggested principles for designing a teaching sequence for the development of one of those concepts in a continuous manner. Although focused on exponentiation, the importance of the work might be beyond a particular content area, and perhaps also it can serve as an example of how other continuous concepts might be thought of in such fields as calculus and operations in algebra.

Moreover, the approach laid out here might also serve as basis for designing the development of continuous concepts in general. The value of using two different theoretical frameworks is revealed through the examination of the hypothetical learning sequence in which the student on the one hand begins with the general image of exponentiation, as suggested by Davydov, coming from a socio-cultural perspective, but at the same time, develops the understanding which is explained from a constructivist perspective. The use of one framework as the leading one for the overall design of the sequence and another framework to design the activities within the sequence might have applications in other conceptual areas.

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